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## CFE METHOD – QUALITY ANALYSIS OF THE APPROXIMATION OF REVERSE LAPLACE TRANSFORM OF FRACTIONAL ORDER

**Summary:** The paper presents the CFE (Continued fraction expansion) method, which allows determination of inverse transform of expression containing element  $s^\alpha$ , by expanding it into a continued fraction. On the basis of the presented method, analysis of selected electrical circuits containing quasi-elements described with fractional derivatives was performed. Calculations and numerical simulations were also carried out.

**Keywords:** reverse transform, electrical circuit, fractional derivative, fractional difference approximations.

## METODA CFE – ANALIZA JAKOŚCI APROKSYMACJI ODWROTNEJ TRANSFORMATY LAPLACE’A UŁAMKOWEGO RZĘDU

**Streszczenie:** W pracy przedstawiono metodę CFE (Continued fraction expansion) umożliwiającą wyznaczenie transformaty odwrotnej wyrażenia zawierającego czynnik  $s^\alpha$  dzięki rozwinięciu w ułamek łańcuchowy. W oparciu o prezentowaną metodę przeprowadzono analizę wybranych obwodów elektrycznych zawierających quasi-elementy opisane pochodnymi niecałkowitego rzędu. Przeprowadzono obliczenia i wykonano symulacje numeryczne.

**Słowa kluczowe:** transformata odwrotna, obwód elektryczny, pochodna ułamkowego rzędu, aproksymacja pochodnej ułamkowej.

### 1. INTRODUCTION

Using Laplace transform method to solve equations containing derivatives of fractional order leads to calculation of inverse transform of expressions containing element  $s^\alpha$ . These inverse transforms may be determined by many different methods, such as Mittag-Leffler function used in control theory [2, 3, 9], Oustaloup approximation method [8], Carlson’s method derived from a regular Newton process [1], Matsuda approximation (this method provides continuous approximation by calculating gain at logarithmically spaced frequencies) [7], and others. In the current work we have applied continued fraction expansion method (CFE) [5, 6], which appears most feasible.

## 2. CFE METHOD

CFE method is based upon expanding expression  $(1+x)^\alpha$  for  $0 \leq \alpha \leq 1$  into continued fraction:

$$(1+x)^\alpha = \frac{1}{1 - \frac{\alpha x}{1 + \frac{(1+\alpha)x}{2 + \frac{(1-\alpha)x}{3 + \frac{(2+\alpha)x}{2 + \frac{(2-\alpha)x}{5 + \dots}}}}} \quad (1)$$

Using notation proposed in [4], Eq. (1) may be written as:

$$(1+x)^\alpha = \frac{1}{1-} \cdot \frac{\alpha x}{1+} \cdot \frac{(1+\alpha)x}{2+} \cdot \frac{(1-\alpha)x}{3+} \cdot \frac{(2+\alpha)x}{2+} \cdot \frac{(2-\alpha)x}{5+} \dots \quad (2)$$

Substituting  $x = s-1$  and taking into account successive terms, we obtain approximation with accuracy to required order. According to [6], up to fifth order these approximations will be equal to:

– first order approximation:

$$s^\alpha \cong \frac{(1+\alpha)s + (1-\alpha)}{(1-\alpha)s + (1+\alpha)} \quad (3)$$

– second order approximation:

$$s^\alpha \cong \frac{(\alpha^2 + 3\alpha + 2)s^2 + (-2\alpha^2 + 8)s + (\alpha^2 - 3\alpha + 2)}{(\alpha^2 - 3\alpha + 2)s^2 + (-2\alpha^2 + 8)s + (\alpha^2 + 3\alpha + 2)} \quad (4)$$

– approximation of higher order may be expressed as follows:

$$s^\alpha \cong \frac{\sum_{k=0}^A P_{Ak}(\alpha) s^{A-k}}{\sum_{k=0}^A Q_{Ak}(\alpha) s^{A-k}} \quad (5)$$

where: A – approximation order,  $P_{Ak}(\alpha)$ ,  $Q_{Ak}(\alpha)$  – polynomials  $\alpha$  of A order.

Hence, for third order A = 3:

$$\begin{aligned} P_{30} = Q_{33} &= \alpha^3 + 6\alpha^2 + 11\alpha + 6 \\ P_{31} = Q_{32} &= -3\alpha^3 - 6\alpha^2 + 27\alpha + 54 \\ P_{32} = Q_{31} &= 3\alpha^3 - 6\alpha^2 - 11\alpha + 54 \\ P_{33} = Q_{30} &= -\alpha^3 + 6\alpha^2 - 11\alpha + 6 \end{aligned} \quad (6)$$

– for fourth order  $A = 4$ :

$$\begin{aligned}
 P_{40} = Q_{44} &= \alpha^4 + 10\alpha^3 + 35\alpha^2 + 50\alpha + 24 \\
 P_{41} = Q_{43} &= -4\alpha^4 - 20\alpha^3 + 40\alpha^2 + 320\alpha + 384 \\
 P_{42} = Q_{42} &= 6\alpha^4 - 150\alpha^3 + 864 \\
 P_{43} = Q_{41} &= -4\alpha^4 + 20\alpha^3 + 40\alpha^2 - 320\alpha + 384 \\
 P_{44} = Q_{40} &= \alpha^4 - 10\alpha^3 + 35\alpha^2 - 50\alpha + 24
 \end{aligned} \tag{7}$$

– for fifth order  $A = 5$ :

$$\begin{aligned}
 P_{45} = Q_{55} &= -a^5 - 15a^4 + 85a^3 - 225a^2 - 274a - 120 \\
 P_{55} = Q_{50} &= a^5 - 15a^4 + 85a^3 - 225a^2 + 274a - 120 \\
 P_{51} = Q_{54} &= 5a^5 + 45a^4 + 5a^3 - 1005a^2 - 3250a - 3000 \\
 P_{54} = Q_{51} &= -5a^5 + 45a^4 - 5a^3 - 1005a^2 + 3250a - 3000 \\
 P_{52} = Q_{52} &= -10a^5 - 30a^4 + 410a^3 - 1230a^2 - 4000a - 12000 \\
 P_{52} = Q_{53} &= 10a^5 - 30a^4 - 410a^3 - 1230a^2 + 4000a - 12000
 \end{aligned} \tag{8}$$

Taking advantage of [4], we have worked out original approximations of higher orders. If Eq.(2) is generalized into following form:

$$(1+x)^\alpha = \frac{1}{1-} \cdot \frac{ax}{1+} \cdot \frac{(1+a)x}{2+} \cdot \frac{(1-a)x}{3+} \dots \frac{(n+a)x}{2+} \cdot \frac{(n-a)x}{(2n+1)+} \dots \tag{9}$$

and when this relationship is written out for e.g. sixth order, then continued fraction assumes following form:

$$\begin{aligned}
 (1+x)^\alpha &= \frac{1}{1-} \cdot \frac{ax}{1+} \cdot \frac{(1+\alpha)x}{2+} \cdot \frac{(1-\alpha)x}{3+} \cdot \frac{(2+\alpha)x}{2+} \cdot \frac{(2-\alpha)x}{5+} \cdot \frac{(3+\alpha)x}{2+} \cdot \frac{(3-\alpha)x}{7+} \\
 &\quad \cdot \frac{(4+\alpha)x}{2+} \cdot \frac{(4-\alpha)x}{9+} \cdot \frac{(5+\alpha)x}{2+} \cdot \frac{(5-\alpha)x}{11+} \cdot \frac{(6-\alpha)x}{2}
 \end{aligned} \tag{10}$$

while polynomials in Eq.(5) and for  $A = 6$  may be written out as:

$$\begin{aligned}
 P_{60} = Q_{66} &= a^6 + 21a^5 + 175a^4 + 735a^3 + 1624a^2 + 1764a - 720 \\
 P_{61} = Q_{65} &= -6a^6 - 84a^5 - 210a^4 + 2100a^3 + 14616a^2 + 33264a - 25920 \\
 P_{62} = Q_{64} &= 15a^6 + 105a^5 - 735a^4 - 6405a^3 + 2520a^2 + 94500a + 162000 \\
 P_{63} = Q_{63} &= -20a^6 + 1540a^4 - 37520a^2 + 288000 \\
 P_{64} = Q_{62} &= 15a^6 - 105a^5 - 735a^4 + 6405a^3 + 2520a^2 + 94500a - 162000 \\
 P_{65} = Q_{61} &= -6a^6 + 84a^5 - 210a^4 - 2100a^3 + 14616a^2 - 33264a + 25920 \\
 P_{66} = Q_{60} &= a^6 - 21a^5 + 175a^4 - 735a^3 + 1624a^2 - 1764a + 720
 \end{aligned} \tag{11}$$

The successive approximations are:

– for  $A = 7$

$$\begin{aligned}
P_{70} = Q_{77} &= -a^7 - 28a^6 - 322a^5 - 1960a^4 - 6769a^3 - 13132a^2 - 13068a - 5040 \\
P_{71} = Q_{76} &= 7a^7 + 140a^6 + 742a^5 - 2800a^4 - 45857a^3 - 193900a^2 - 358092a - 2466960 \\
P_{72} = Q_{75} &= -21a^7 - 252a^6 + 798a^5 + 20160a^4 + 46851a^3 - 337428a^2 - 1741068a - 2222640 \\
P_{73} = Q_{74} &= 35a^7 + 140a^6 - 3850a^5 - 15400a^4 + 136115a^3 + 544460a^2 - 1543500a - 6174000 \quad (12) \\
P_{74} = Q_{73} &= -35a^7 + 140a^6 + 3850a^5 - 15400a^4 - 136115a^3 + 544460a^2 - 1543500a - 6174000 \\
P_{75} = Q_{72} &= 21a^7 - 252a^6 - 798a^5 + 20160a^4 - 46851a^3 - 337428a^2 + 1741068a - 2222640 \\
P_{76} = Q_{71} &= -7a^7 + 140a^6 - 742a^5 - 2800a^4 + 45857a^3 - 193900a^2 + 358092a - 2466960 \\
P_{77} = Q_{70} &= a^7 - 28a^6 + 322a^5 - 1960a^4 + 6769a^3 - 13132a^2 + 13068a - 5040
\end{aligned}$$

– for  $A = 8$

$$\begin{aligned}
P_{89} = Q_{88} &= a^8 + 36a^7 + 546a^6 + 4536a^5 + 22449a^4 + 67284a^3 + 118124a^2 + 109584a + 40320 \\
P_{81} = Q_{87} &= -8a^8 - 216a^7 - 1848a^6 + 504a^5 + 110208a^4 + 788256a^3 + 2572928a^2 + 4110336a + 2580480 \\
P_{82} = Q_{86} &= 28a^8 + 504a^7 + 168a^6 - 47376a^5 - 278628a^4 + 498456a^3 + 9310112a^2 + 30030336a + 31610880 \\
P_{83} = Q_{85} &= -56a^8 - 504a^7 + 7224a^6 + 75096a^5 - 236544a^4 - 3630816a^3 - 1746304a^2 + 56899584a + 126443520 \\
P_{84} = Q_{84} &= 70a^8 - 12180a^6 + 765030a^4 - 20509720a^2 + 197568000 \\
P_{85} = Q_{83} &= 56a^8 + 504a^7 + 7224a^6 - 75096a^5 - 236544a^4 + 3630816a^3 - 1746304a^2 - 56899584a + 126443520 \\
P_{86} = Q_{82} &= 28a^8 - 504a^7 + 168a^6 + 47376a^5 - 278628a^4 - 498456a^3 + 9310112a^2 - 30030336a + 31610880 \\
P_{87} = Q_{81} &= -8a^8 + 216a^7 - 1848a^6 - 504a^5 + 110208a^4 - 788256a^3 + 2572928a^2 - 4110336a + 2580480 \\
P_{88} = Q_{80} &= a^8 - 36a^7 + 546a^6 - 4536a^5 + 22449a^4 - 67284a^3 + 118124a^2 - 109584a + 40320 \quad (13)
\end{aligned}$$

– for  $A = 9$

$$\begin{aligned}
P_{90} = Q_{99} &= -a^9 - 45a^8 - 870a^7 - 9450a^6 - 63273a^5 - 269325a^4 + 723680a^3 - \\
&\quad - 1172700a^2 + 1026576a - 362880 \\
P_{91} = Q_{98} &= -9a^9 + 315a^8 - 3870a^7 + 10710a^6 + 206703a^5 - 2494485a^4 + 12807720a^3 - \\
&\quad - 35256060a^2 + 50493456a - 29393280 \\
P_{92} = Q_{97} &= 36a^9 - 900a^8 + 3600a^7 + 88200a^6 - 993132a^5 + 1377180a^4 + 31347000a^3 - \\
&\quad - 206128800a^2 + 513962496a - 470292480 \\
P_{93} = Q_{96} &= 84a^9 + 1260a^8 - 10080a^7 + 234360a^6 - 173628a^5 - 13528620a^4 + 55051080a^3 - \\
&\quad - 196187040a^2 - 1578963456a - 2560481280 \\
P_{94} = Q_{95} &= -126a^9 - 630a^8 + 28980a^7 + 144900a^6 - 24283398a^5 - 12141990a^4 + 87748920a^3 + \\
&\quad + 438744600a^2 - 1152216576a - 5761082880 \\
P_{95} = Q_{94} &= 126a^9 - 630a^8 - 28980a^7 + 144900a^6 + 24283398a^5 - 12141990a^4 - 87748920a^3 + \\
&\quad + 438744600a^2 + 1152216576a - 5761082880 \\
P_{96} = Q_{93} &= -84a^9 + 1260a^8 + 10080a^7 - 234360a^6 + 173628a^5 + 13528620a^4 - 55051080a^3 - \\
&\quad - 196187040a^2 + 1578963456a - 2560481280 \\
P_{97} = Q_{92} &= 36a^9 - 900a^8 + 3600a^7 + 88200a^6 - 993132a^5 + 1377180a^4 + 31347000a^3 - \\
&\quad - 206128800a^2 + 513962496a - 470292480 \\
P_{98} = Q_{91} &= -9a^9 + 315a^8 - 3870a^7 + 10710a^6 + 206703a^5 - 2494485a^4 + 12807720a^3 - \\
&\quad - 35256060a^2 + 50493456a - 29393280 \\
P_{99} = Q_{90} &= a^9 - 45a^8 + 870a^7 - 9450a^6 + 63273a^5 - 269325a^4 + 723680a^3 - \\
&\quad - 1172700a^2 + 1026576a - 362880 \quad (14)
\end{aligned}$$

– for  $A = 10$

$$\begin{aligned}
P_{100} = Q_{100} &= a^{10} + 55a^9 + 1320a^8 + 18150a^7 + 157773a^6 + 902055a^5 + 3416930a^4 + 8409500a^3 + \\
&\quad + 12753576a^2 + 10628640a + 362880 \\
P_{101} = Q_{109} &= -10a^{10} - 440a^9 - 7260a^8 - 42240a^7 + 279510a^6 + 6477240a^5 + 49558960a^4 + 208039040a^3 + \\
&\quad + 505375200a^2 + 663696000a - 36288000 \\
P_{102} = Q_{108} &= 45a^{10} + 1485a^9 + 11880a^8 - 127710a^7 - 2713095a^6 - 12401235a^5 + 64115370a^4 + 942731460a^3 + \\
&\quad + 4265929800a^2 + 8949204000a - 7348320000 \\
P_{103} = Q_{107} &= -120a^{10} - 2640a^9 + 7920a^8 + 570240a^7 + 279920a^6 - 33208560a^5 - 335908320a^4 - 13841040a^3 + \\
&\quad + 9151084800a^2 + 39688704000a + 52254720000 \\
P_{104} = Q_{106} &= 210a^{10} + 2310a^9 - 554400a^8 - 679140a^7 + 4802490a^6 + 73201590a^5 - 11177160a^4 - 3425535060a^3 - \\
&\quad - 4008034800a^2 + 58677696000a + 1600300800 \\
P_{105} = Q_{105} &= -252a^{10} - 83160a^8 - 10652796a^6 + 66117740a^4 - \\
&\quad + 19854217152a^2 + 230443315200 \\
P_{106} = Q_{104} &= 210a^{10} - 2310a^9 - 554400a^8 + 679140a^7 + 4802490a^6 - 73201590a^5 - 11177160a^4 + 3425535060a^3 - \\
&\quad - 4008034800a^2 - 58677696000a + 1600300800 \\
P_{107} = Q_{103} &= -120a^{10} + 2640a^9 + 7920a^8 - 570240a^7 + 279920a^6 + 33208560a^5 - 335908320a^4 + 13841040a^3 + \\
&\quad + 9151084800a^2 - 39688704000a + 52254720000 \\
P_{108} = Q_{102} &= 45a^{10} - 1485a^9 + 11880a^8 + 127710a^7 - 2713095a^6 + 12401235a^5 + 64115370a^4 - 942731460a^3 + \\
&\quad + 4265929800a^2 - 8949204000a + 7348320000 \\
P_{109} = Q_{101} &= -10a^{10} + 440a^9 - 7260a^8 + 42240a^7 + 279510a^6 - 6477240a^5 + 49558960a^4 - 208039040a^3 + \\
&\quad + 505375200a^2 - 663696000a - 36288000 \\
P_{110} = Q_{100} &= a^{10} - 55a^9 + 1320a^8 - 18150a^7 + 157773a^6 - 902055a^5 + 3416930a^4 - 8409500a^3 + \\
&\quad + 12753576a^2 - 10628640a + 362880
\end{aligned} \tag{15}$$

It must be pointed out, that polynomials present in approximations contain integral powers of  $s$ , and therefore in order to determine inverse transform we are able to apply well-known and standard methods.

### 3. COMPARISON OF ACCURACY FOR DIFFERENT ORDER APPROXIMATIONS

In order to evaluate and select appropriately accurate approximation of possibly least order, a series of numerical experiments has been run. These experiments have consisted of comparing different order approximations with approximated function, i.e.  $s^\alpha$ . Percentage errors of approximation (order 3 to 8) for three selected  $\alpha$  values:  $\alpha = 0.5$ ,  $0.8$  and  $0.9$  are shown in Figs. 1, 2 and 3.

Looking at these diagrams, we may arrive at a conclusion that approximation of 7<sup>th</sup> order is most accurate. Approximations of orders higher than seventh are less accurate; this is possibly due to large number of necessary calculations and cumulation of numerical errors in the computations – see Eqs. (12), (13) and (14).

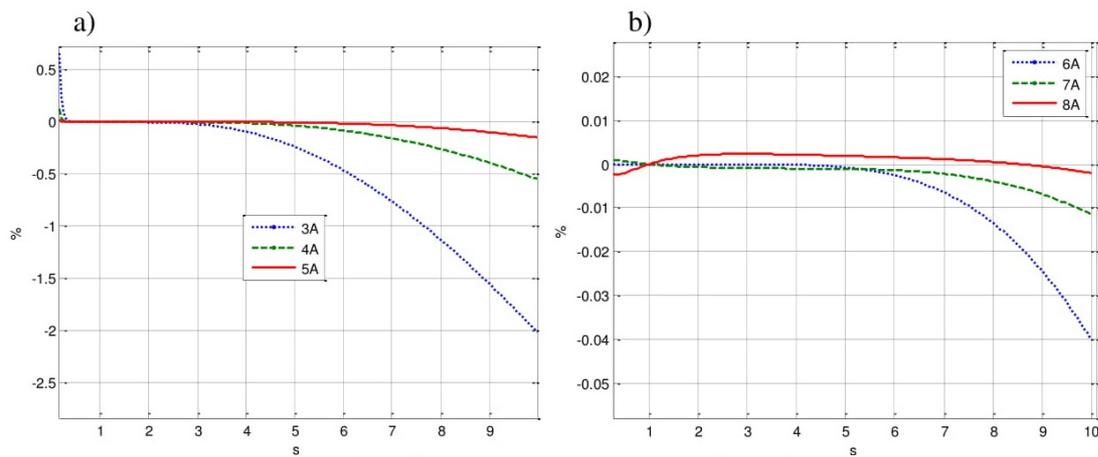


Fig.1. Approximation errors of  $s^\alpha$  curve,  $\alpha = 0.5$ ; a) orders 3,4 and 5, b) orders 6, 7 and 8  
 Rys.1 Błędy aproksymacji przebiegu  $s^\alpha$ ,  $\alpha = 0.5$ ; a) rzędu 3,4 i 5; b) rzędu 6,7 i 8

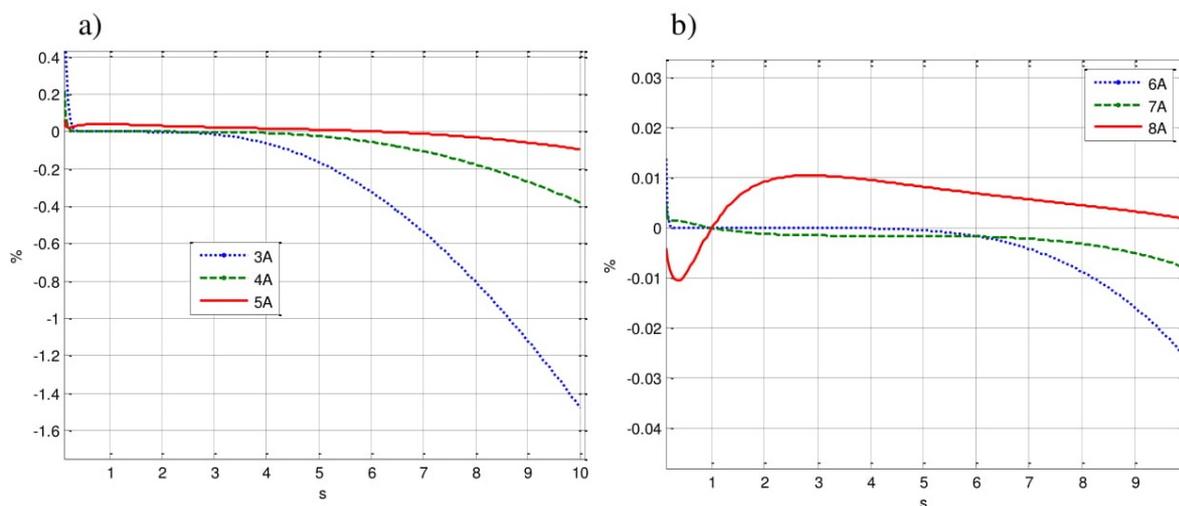


Fig.2. Approximation errors of  $s^\alpha$  curve,  $\alpha = 0.8$ ; a) orders 3,4 and 5, b) orders 6, 7 and 8  
 Rys.2. Błędy aproksymacji przebiegu  $s^\alpha$ ,  $\alpha = 0.8$ . a) rzędu 3, 4 i 5; b) rzędu 6,7 i 8

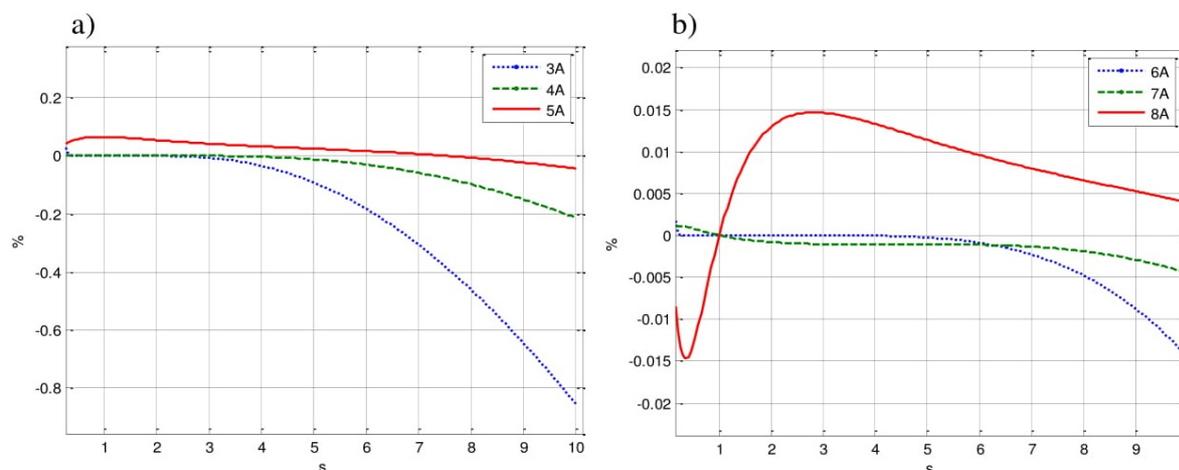


Fig.3. Approximation errors of  $s^\alpha$  curve,  $\alpha = 0.9$ ; a) orders 3,4 and 5, b) orders 6, 7 and 8  
 Rys.3. Błędy aproksymacji przebiegu  $s^\alpha$ ,  $\alpha = 0.9$ . a) rzędu 3, 4 i 5; b) rzędu 6,7 i 8

#### 4. EXAMPLE OF $RL^\alpha$ CIRCUIT WITH SINUSOIDAL SUPPLY SWITCHED ON

We shall discuss  $RL^\alpha$  circuit containing resistor and coil (inductance) connected in series. Coil is described with the relationship  $u(t) = L D^\alpha i(t)$ , where  $D^\alpha$  denotes fractional derivative of order  $\alpha$  in accordance with Caputo definition [10]. Circuit is switched on and supplied with sinusoidal voltage. Equation for this circuit may be expressed as:

$$D^\alpha i(t) + Ri(t) = U \sin \omega t \quad (16)$$

If initial condition is set at zero and when (16) is subjected to Laplace transform, we obtain:

$$Ls^\alpha I(s) + RI(s) = \frac{U\omega}{s^2 + \omega^2} \quad (17)$$

Then we may determine current transform as:

$$I(s) = \frac{U\omega}{(s^2 + \omega^2)(Ls^\alpha + R)} \quad (18)$$

When a selected approximation in the form of fraction  $L_A(s)/M_A(s)$  is substituted for  $s^\alpha$ , we obtain:

$$I(s) = \frac{U\omega}{(s^2 + \omega^2) \left( L \frac{L_A(s, \alpha)}{M_A(s, \alpha)} + R \right)} \quad (19)$$

After some elementary operations we get:

$$I(s) = \frac{U\omega M_A(s, \alpha)}{(s^2 + \omega^2)(L \cdot L_A(s, \alpha) + R M_A(s, \alpha))} \quad (20)$$

Determination of inverse transform of (20), i.e. current versus time waveform requires information on poles in (20). Number of these poles is equal to order of approximation used plus two (conjugate poles of factor  $(s^2 + \omega^2)$ ). In the investigated cases the poles have been simple and this has facilitated determination of inverse transform.

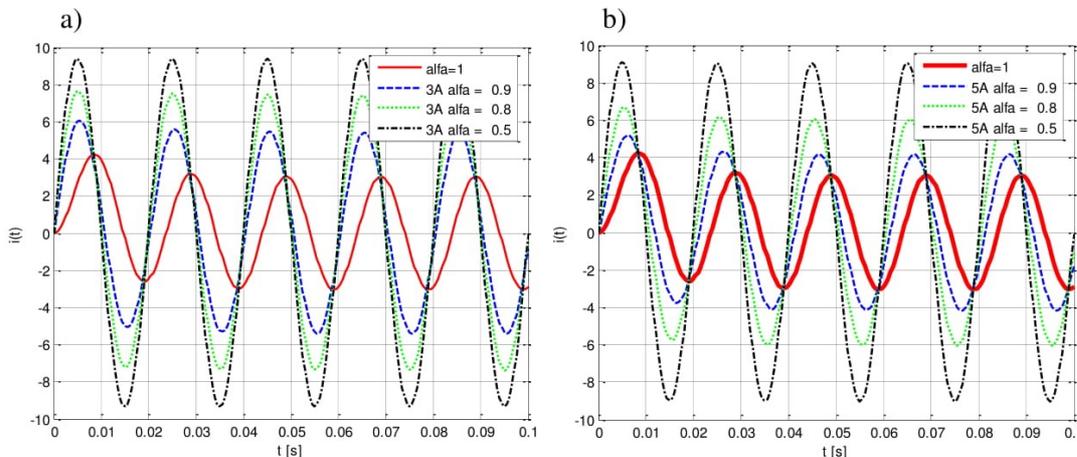


Fig.4. Current waveforms in  $RL^\alpha$  circuit for approximations of: a) 4<sup>th</sup> order, b) 5<sup>th</sup> order  
Rys.4 Przebiegi prądu w obwodzie  $RL^\alpha$  dla aproksymacji: a) 4 rzędu, b) 5 rzędu

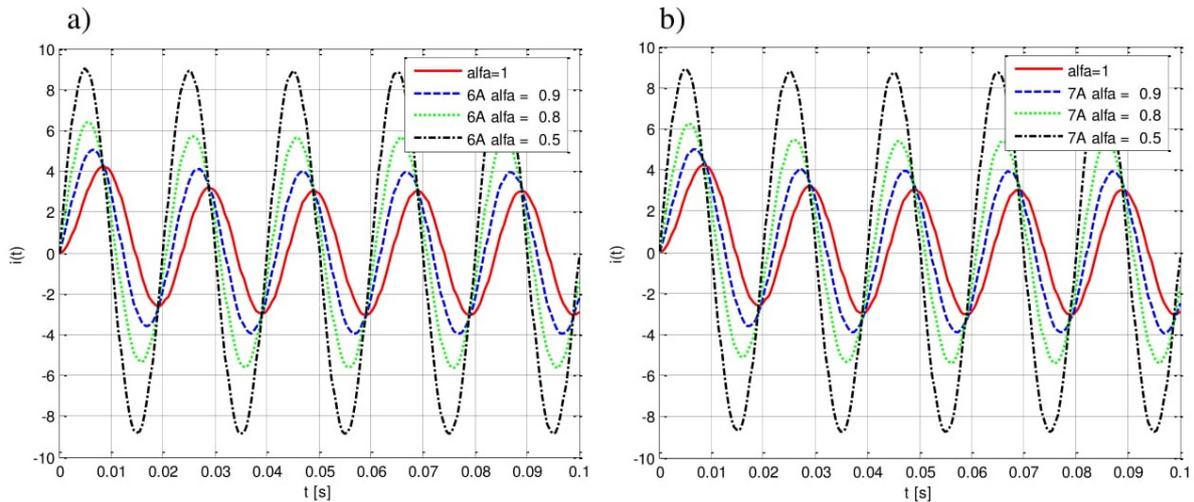


Fig.5. Current waveforms in  $RL^\alpha$  circuit for approximations of: a) 6<sup>th</sup> order, b) 7<sup>th</sup> order

Rys.5 Przebiegi prądu w obwodzie  $RL^\alpha$  dla aproksymacji: a) 6 rzędu, b) 7 rzędu

Analysing waveforms shown in Figs. 5 and 6, we may conclude that they do not greatly differ from each other. Hence, it may be inferred that it is not worthwhile to apply approximations of higher orders since the calculations will be very tedious (with increasing number of poles, number of solution components also increases). It seems that approximation of 5<sup>th</sup>, 6<sup>th</sup> or 7<sup>th</sup> order will be quite sufficient.

## 5. CONCLUSIONS

Conventional (standard) approach to analysis of electrical circuits and their modelling usually ignores effects associated with existence of non-ideal elements (lossiness, non-linearity). Reliable models may not always be obtained with this type of approach; it may lead to errors in designing particular types of circuits. Use of fractional order derivatives helps us to “compensate” these neglected phenomena in discussed systems.

Application of CFE method for determination of inverse transform, i.e. solving of differential equation of fractional order has proved to be most interesting. Using approximations of increasing orders, we obtained more and more accurate solutions. However, this was achieved at the cost of increasingly complex calculations (polynomials of higher order, i.e. greater number of poles). When attained approximations and their errors are analysed, we may deduce that optimum approximation lies somewhere between order 5 and 7. Application of higher order approximations is not necessary since calculations will be very tedious (with increasing number of poles, number of solution components also increases).

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