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## PROPAGATION OF RECTANGULAR PULSES IN THE CASCADE OF FOUR-TERMINAL NETWORKS

**Summary:** The paper presents a method of determining the transmittance of circuits with distributed parameters, in this case a transmission line consisting of a cascade of identical four-terminal networks. To construct the transmittance, signal flow graph theory was used. The result is provided in the form of a rational function which has two single poles and four multiple poles  $((N-2)$ -tuple poles), if network consists of  $N$  four-terminal networks loaded with resistance. On the basis of the transmittance we calculated load voltage with unit step function or rectangular pulse at the input. Conclusions on transmission possibilities for digital signals have been formulated on the basis of obtained results.

**Keywords:** transient states, transfer function, cascade of identical four-terminal networks,

## PROPAGACJA IMPULSÓW PROSTOKĄTNYCH W KASKADZIE CZWÓRNIKÓW

**Streszczenie:** W pracy przedstawiono metodę wyznaczania transmitancji operatorowej obwodów o parametrach rozłożonych – linii długiej w oparciu o kaskadę jednakowych czwórników. Do konstrukcji tej transmitancji wykorzystano teorię grafów przepływu sygnałów. W wyniku otrzymano wyrażenie w postaci funkcji wymiernej, która posiada dwa bieguny jednokrotne i cztery bieguny  $N-2$  krotne przy  $N$  czwórnikach obciążonych rezystancją. Na podstawie transmitancji obliczono przebieg napięcia na obciążeniu przy wymuszeniu funkcją jednostkową i impulsem prostokątnym. Uzyskane wyniki pozwoliły na sformułowanie wniosków dotyczących możliwości transmisyjnych dla sygnałów cyfrowych..

**Słowa kluczowe:** stany nieustalone, transmitancja, kaskada jednakowych czwórników.

### 1. INTRODUCTION

The issue of propagation of rectangular pulses in cascade-connected four-terminal networks is strictly related to analysis of transient states in transmission lines. This problem has been discussed in relatively few publications. Solution of telegraph equations by using

finite difference method together with Fehlberg method has been proposed in [1]. Two-dimensional Laplace transformation has been applied in [2,4], but to lossless lines only. Method of finite differences in distance and time domain as well as time and frequency domain has been used for lossless lines in [3]. Investigation of transient states of signal propagation in transmission line has been discussed in [6]; problem has been examined from the viewpoint of utilizing the results for identifying location of damage in the line. Transmission line has been subjected to differently shaped input signals and alterations in waveform shape have been analysed after signals had passed through a definite number of four-terminal networks. In current paper we have used transfer function for cascade consisting of identical four-terminals networks, characterised by lateral and longitudinal loss elements both; this transfer function has been determined earlier (see [7,8]).

## 2. TRANSFER FUNCTION OF FOUR-TERMINAL NETWORK CASCADE

The following formula for transfer function of  $n$  identical “G-type” four-terminal networks has been presented in [8]:

$$T(s) = \frac{[(R_0 + sL_0)(G_0 + sG_0) + 2]^{n-3}}{\left\{[(R_0 + sL_0)(G_0 + sG_0) + 2]^2 - 1\right\}^{n-2} (R_0 + sL_0)(G_0 + sG_0) + 1} \quad (1)$$

where:  $R_0$ ,  $L_0$ ,  $G_0$ ,  $C_0$  – parameters of single four-terminal network (p.u. parameters of transmission line),  $G_2$  – load conductance.

When inverse transform is calculated, poles of transform (1) are of tremendous importance. Transform has two single poles for  $(R_0 + sL_0)(G_2 + G_0 + sC_0) + 1 = 0$ :

$$\begin{aligned} s_5 &= \frac{-(G_0 + G_2)L_0 - C_0R_0 - \sqrt{((G_0 + G_2)L_0 + C_0R_0)^2 - 4C_0L_0(1 + (G_0 + G_2)R_0)}}{2C_0L_0} \\ s_6 &= \frac{-(G_0 + G_2)L_0 - C_0R_0 + \sqrt{((G_0 + G_2)L_0 + C_0R_0)^2 - 4C_0L_0(1 + (G_0 + G_2)R_0)}}{2C_0L_0} \end{aligned} \quad (2)$$

and four multiple poles ((n-2)-tuple) for  $[(R_0 + sL_0)(G_0 + sC_0) + 2]^2 - 1 = 0$

$$\begin{aligned} s_1 &= \frac{-G_0L_0 - C_0R_0 - \sqrt{(G_0L_0 + C_0R_0)^2 - 4C_0L_0(1 + G_0R_0)}}{2C_0L_0} \\ s_2 &= \frac{-G_0L_0 - C_0R_0 + \sqrt{(G_0L_0 + C_0R_0)^2 - 4C_0L_0(1 + G_0R_0)}}{2C_0L_0} \\ s_3 &= \frac{-G_0L_0 - C_0R_0 - \sqrt{(G_0L_0 + C_0R_0)^2 - 4C_0L_0(3 + G_0R_0)}}{2C_0L_0} \\ s_4 &= \frac{-G_0L_0 - C_0R_0 + \sqrt{(G_0L_0 + C_0R_0)^2 - 4C_0L_0(3 + G_0R_0)}}{2C_0L_0} \end{aligned} \quad (3)$$

Analysing expressions (2) and (3) we may easily arrive at a conclusion that all poles will be real. This means that there will be no oscillations in the circuit, when for single poles

$$((G_0 + G_2)L_0 + C_0R_0)^2 > 4C_0L_0(1 + (G_0 + G_2)R_0) \quad (4)$$

and for multiple poles:

$$(G_0L_0 + C_0R_0)^2 > 4C_0L_0(1 + G_0R_0) \quad (5)$$

If we assume that leakage conductance is very small ( $G_0 \approx 0$ ), and load resistance is very high ( $G_2 \approx 0$ ), then all poles will be real if  $R_0 > 2\sqrt{\frac{L_0}{C_0}}$ .

Unfortunately, in case of power transmission and telecommunication lines this condition is not fulfilled, so that poles will be complex and conjugate in pairs, with negative real parts.

In order to determine inverse transform of transfer function (1), we have used partial fraction expansion [5]. In spite of the fact that this method is very simple since it is established upon well-known algebraic decomposition of rational function into partial fractions and calculation of inverse transform separately for each fraction, its interpretation may be shown basing on Laurent series theory, while fraction coefficients are equivalent to residue of function (1) in the neighbourhood of singular points (i.e. poles).

Hence, transfer function (1) may be presented as sum of partial factors [5]

$$T(s) = \frac{L(s)}{N(s)} = \frac{C_5}{s - s_5} + \frac{C_6}{s - s_6} + \sum_{i=1}^4 \sum_{k=1}^{n-2} \frac{C_{ik}}{(s - s_i)^k} \quad (6)$$

where coefficients  $C_{ik}$  are expressed as

$$C_{ik} = \frac{1}{(n-3)!} \lim_{s \rightarrow s_i} \frac{d^{n-2-k}}{ds^{n-2-k}} \left[ \frac{L(s)}{N(s)} (s - s_i)^{n-2} \right] \quad (7)$$

Coefficients  $C_5$  and  $C_6$  relate to single poles and may therefore be calculated with the help of Heaviside expansion formula:

$$C_j = \frac{L(s_j)}{N'(s_j)}, \quad j = 5, 6 \quad (8)$$

Finally, inverse transform of transfer function (1) is equal to:

$$L^{-1}\{T(s)\} = C_5 e^{s_5 t} + C_6 e^{s_6 t} + \sum_{i=1}^4 \sum_{k=1}^{n-2} \frac{C_{ik}}{(k-1)!} t^{k-1} e^{s_i t} \quad (9)$$

In order to calculate circuit response to input function expressed with  $U(s)$  transform, we must calculate inverse transform of the product  $U(s)T(s)$ ; this means in practice that for dc input we must introduce additional pole  $s_7 = 0$ , and in case of sinusoidal input two conjugated poles must be added  $s_{7,8} = \pm j\omega$ .

### 3. RECTANGULAR FUNCTION INPUT

Input function in the form of single rectangular pulse with amplitude  $A$  and width  $a > 0$  (step function) may be expressed as:

$$u(t) = A\mathbf{1}(t) - A\mathbf{1}(t-a) \quad (10)$$

and its transform is:

$$U(s) = \frac{A}{s} - \frac{A}{s} e^{-as} \quad (11)$$

Voltage transform at the end terminals of cascade consisting of  $n$  identical four-terminal networks is equal to:

$$U_2(s) = \frac{[(R_0 + sL_0)(G_0 + sC_0) + 2]^{n-3}}{\{[(R_0 + sL_0)(G_0 + sC_0) + 2]^2 - 1\}^{n-2}} \frac{A}{s(R_0 + sL_0)(G_2 + G_0 + sC_0) + 1} (1 - e^{-as}) \quad (12)$$

where factor  $e^{-as}$  is shift in time domain by value  $a$ .

Inverse transform of this voltage, i.e. voltage versus time waveform (12) is equal to:

$$u_2(t) = C_5 (e^{s_5 t} - e^{s_5 (t-a)}) + C_6 (e^{s_6 t} - e^{s_6 (t-a)}) + \sum_{i=1}^4 \sum_{k=1}^{n-2} \frac{C_{ik}}{(k-1)!} [t^{k-1} e^{s_i t} - (t-a)^{k-1} e^{s_i (t-a)}] \quad (13)$$

Equations (7) and (8) are still true when  $L_I(s) = AL(s)$  is substituted for  $L(s)$  and  $N_I(s) = s N(s)$  is substituted for  $N(s)$ .

### 4. NUMERICAL CALCULATIONS

Numerical simulations have been run in MATLAB environment, programs have been invented by authors of this work.

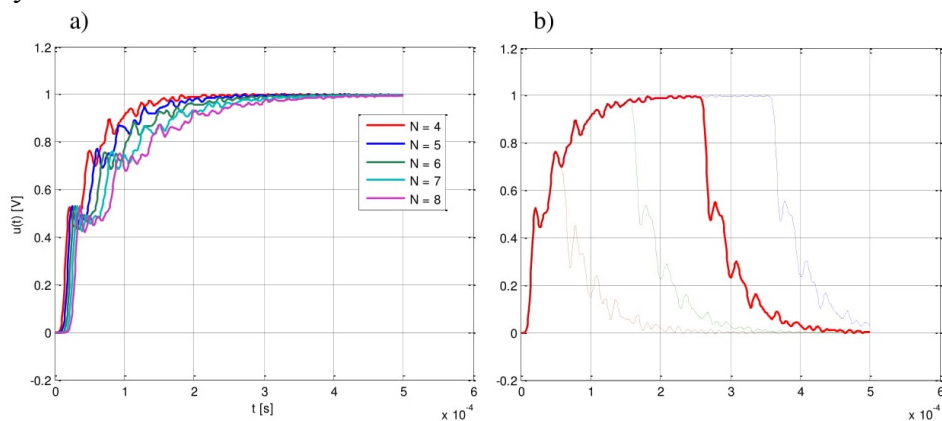


Fig.1. Voltage course at end terminals of the cascade constructed of  $N=4-8$  four-terminals network  
a) Heaviside step function input, b) rectangular function input with  $a = 0.5, 1.5, 2.5, 3.5 \cdot 10^{-4}$  s for  $N = 8$

Rys.1 Przebieg napięcia na końcu kaskady przy wymuszeniu przy  $N=4-8$  czwórnikach:  
a) skokiem jednostkowym, b) impulsem prostokątnym o szerokości  $a = 0.5, 1.5, 2.5, 3.5 \cdot 10^{-4}$  s dla  $N = 8$

Determination of  $C_{ik}$  coefficients (7) has proved to be most difficult since derivatives of ratio  $L(s)/N(s)$  in symbolic form had to be calculated. MATLAB environment makes such calculations possible, but for derivatives of higher orders ( $> 6$ ) calculation time increases considerably. That is why results presented here relate to limited number of four-terminal networks only ( $N=8$ ). Analysing waveforms shown in Fig.1a we may notice that with increasing number of four-network terminals voltage delay at the end terminals of the cascade is increased (this is intuitively understood). This effect bears significant influence on signal received when input function is rectangular (Fig.1b). For a given line we may determine minimum pulse width time, for which output signal may be correctly interpreted. Maximum transmission speed is a consequence of the determined signal width.

## 5. CONCLUSIONS

Analysis of lossy circuits with distributed parameters in dynamic states may be rated amongst most difficult problems in the circuit theory. In majority of cases there is no analytical solution of the problem; the only possible way of obtaining information about current and voltage waveforms is by applying approximate methods of numerical integration. The method presented in current paper is based upon cascade connection of identical four-network terminals and makes it possible to determine transfer function of  $N$  four-terminal networks and then voltage  $U_2(s)$  at end terminals of this circuit provided that transform of input function is known. When inverse transform of this voltage is calculated, we obtain voltage waveform (voltage versus time).

Analysing obtained waveforms, we may easily observe that with increasing number of four-network terminals voltage delay at the end terminals of the cascade is increased. If rectangular input function is characterised by too narrow signal, it may cause faulty interpretation of output signal. Therefore, for a given line we may determine maximum transmission speed of such signals.

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